
Evaluate the definite integrals and find the indefinite integrals. EXACT ANSWERS ONLY PLEASE!!!

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1. $\int_0^5 \sqrt{49-x^2}\,dx$

$= \int_0^{\arcsin 5/7} (7\cos\theta)(7\cos\theta\, d\theta)$

$= \int_0^{\arcsin 5/7} 49\cos^2\theta\, d\theta$

$= 49\int_0^{\arcsin 5/7} \frac{1+\cos 2\theta}{2}\, d\theta$

$= \frac{49}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{\theta=0}^{\theta=\arcsin 5/7}$

$= \frac{49}{2}\left[\left(\arcsin 5/7 + \frac{\sin 2(\arcsin 5/7)}{2}\right) - \left(\overset{0}{\arcsin 0} + \frac{\overset{0}{\sin(2\cdot 0)}}{2}\right)\right]$

$= \frac{49}{2}\left[\arcsin\frac{5}{7} + \frac{2\sin(\arcsin\frac{5}{7})\cos(\arcsin 5/7)}{2}\right]$

$= \frac{49}{2}\left[\arcsin\frac{5}{7} + \left(\frac{5}{7}\right)\cos\left(\arcsin\frac{5}{7}\right)\right]$

$= \frac{49}{2}\left[\arcsin\frac{5}{7} + \frac{5}{7}\cos\left(\arcsin\frac{5}{7}\right)\right]$

$= \frac{49}{2}\left[\frac{5}{7} + \frac{5}{7}\cdot\frac{2\sqrt{6}}{7}\right]$

$= \frac{49}{2}\left(\frac{35+10\sqrt{6}}{49}\right) \rightarrow = \boxed{\frac{5}{2}\left(7+2\sqrt{6}\right)}$

So sorry for the ugly top limit value!

$a=7$

$\arcsin\left(\frac{x}{7}\right) = \theta$

$x = 7\sin\theta$

$\frac{dx}{d\theta} = 7\cos\theta$

$dx = 7\cos\theta\, d\theta$

$\sin\theta = \frac{x}{7}$

$\ast\ \sin 2\theta = 2\sin\theta\cos\theta$

upper limit:

$5 = 7\sin\theta$

$\theta = \arcsin\frac{5}{7}$

lower limit:

$0 = 7\sin\theta$

$\arcsin 0 = \theta$

$0 = \theta$

$\sqrt{49-x^2} = \sqrt{49-(7\sin\theta)^2}$

$= \sqrt{49-49\sin^2\theta}$

$= \sqrt{49(1-\sin^2\theta)}$

$= 7\sqrt{\cos^2\theta}$

$= 7\cos\theta$

note:

$\theta = \frac{5}{7} \in QI$

$\theta = \arcsin x/7$

$\sin\theta = \frac{x}{7}$

$x=5$, so

$\sin(\arcsin\theta) = \frac{5}{7}$

$\cos(\arcsin\theta) = \frac{\sqrt{49-25}}{7} = \frac{\sqrt{24}}{7} = \frac{2\sqrt{6}}{7}$

2. $$\int \frac{\sin^3 3x}{\sqrt{\cos 3x}}\,dx$$

$$= \int (\sin^2 3x)(\sin 3x)(\cos 3x)^{-1/2}\,dx$$

$$= \int (1-\cos^2 x)(\cos 3x)^{-1/2}(\sin 3x)\,dx$$

$$= -\frac{1}{3}\int (\cos 3x)^{-1/2}(\sin 3x\,dx)(-3) - \left(-\frac{1}{3}\right)\int (\cos 3x)^{3/2}(\sin 3x\,dx)(-3)$$

$$= -\frac{1}{3}\cdot\frac{(\cos 3x)^{1/2}}{1/2} + \frac{1}{3}\cdot\frac{(\cos 3x)^{5/2}}{5/2} + C$$

$$= \boxed{-\frac{2}{3}(\cos 3x)^{1/2} + \frac{2}{15}(\cos 3x)^{5/2} + C}$$

Both integrals have

$g(x) = \cos 3x$

$g'(x) = -3\sin 3x$

3. $\int (x+3)\sqrt{1-x}dx = (x+3)\left[-\frac{2}{3}(1-x)^{3/2}\right] - \int \left(-\frac{2}{3}(1-x)^{3/2}\right) dx$

$u = x+3$

$\frac{du}{dx} = 1$

$du = dx$

$\int dv = -\int (1-x)^{1/2} dx$

$v = -\frac{(1-x)^{3/2}}{3/2}$

$v = -\frac{2}{3}(1-x)^{3/2}$

$= -\frac{2}{3}(x+3)(1-x)^{3/2} + -\frac{2}{3}\int -(1-x)^{3/2} dx$

$= -\frac{2}{3}(x+3)(1-x)^{3/2} - \frac{2}{3}\left[\frac{(1-x)^{5/2}}{2/5}\right] + C$

$= \boxed{-\frac{2}{3}(x+3)(1-x)^{3/2} - \frac{5}{3}(1-x)^{5/2} + C}$ I will accept this answer

$= -\frac{1}{3}(1-x)^{3/2}\left[2(x+3) + 5(1-x)^{1}\right] + C$

$= -\frac{1}{3}(1-x)^{3/2}\left[2x+6+5-5x\right] + C$

$= \boxed{-\frac{1}{3}(1-x)^{3/2}(11-3x) + C}$ prettiest answer

4. $\int \tan^3\theta \sec^3\theta d\theta = \int \tan^2\theta \sec^2\theta \sec\theta \tan\theta d\theta$

$= \int (\sec^2\theta - 1)(\sec\theta)(\sec\theta\tan\theta)d\theta$

$= \int (\sec\theta)^3 (\sec\theta\tan\theta)d\theta - \int (\sec\theta)(\sec\theta\tan\theta)d\theta$

$= \boxed{\frac{(\sec\theta)^4}{4} - \frac{(\sec\theta)^2}{2} + C}$

For both integrals,
$g(\theta) = \sec\theta$
$g'(\theta) = \sec\theta\tan\theta$

$$\sin^2 3x = \frac{1-\cos 6x}{2}$$

$$\left.\begin{array}{l} g(x) = 6x \\ g'(x) = 6 \end{array}\right\} \text{2nd integral}$$

5. $\int \sin^4 3x\,dx = \int (\sin^2 3x)^2\,dx$

$$= \int \left(\frac{1-\cos 6x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 6x + \cos^2 6x\right) dx$$

$$= \frac{1}{4}\left[\int 1\,dx - 2\cdot\frac{1}{6}\int \cos 6x\,dx\cdot 6 + \int \frac{1+\cos 12x}{2}\,dx\right]$$

$$= \frac{1}{4}\left[x - \frac{1}{3}\cdot \sin 6x + \frac{1}{2}\left[x + \frac{1}{12}\sin 12x\right]\right] + C$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \frac{1}{3}\sin 6x + \frac{1}{24}\sin 12x\right] + C$$

$$= \boxed{\frac{1}{96}\left(36x - 8\sin 6x + \sin 12x\right) + C}$$

6. $\int \ln 5x\,dx = (\ln 5x)(x) - \int (x)\left(\frac{dx}{x}\right)$

$= x\ln 5x - \int dx$

$= x\ln 5x - x + C$

$= \boxed{x(\ln 5x - 1) + C}$

$u = \ln 5x$

$\frac{du}{dx} = \frac{5}{5x} = \frac{1}{x}$

$du = \frac{dx}{x}$

$\int dv = \int dx$

$v = x$

7. $\int_0^1 \frac{x^2-x}{x^2+x+1}dx$

$= \int_0^1 \left(1-\frac{2x}{x^2+x+1}\right)dx$

$= \int_0^1 1dx - \underbrace{\int_0^1 \frac{2x+1-1}{x^2+x+1}dx}$

$= \int_0^1 1dx - \underbrace{\int_0^1 \frac{2x+1}{x^2+x+1}dx - \int_0^1 \frac{-1}{x^2+x+1}dx}$

$= x\Big|_{x=0}^{x=1} - \ln|x^2+x+1|\Big|_0^1 + \int_0^1 \frac{1}{\left(\sqrt{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right)^2}dx$

$= (1-0)-\left(\ln|1^2+1+1|-\ln|0^2+0+1|\right) + \int_0^1 \frac{dx}{\left(\sqrt{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right)^2}$

$= 1-\ln|3|-\ln|1| + \int_{\arctan\frac{8\sqrt{3}}{3}}^{\arctan\frac{8}{3\sqrt{3}}} \frac{\frac{\sqrt{3}}{2}\cancel{\sec^2}\theta d\theta}{\frac{3}{4}\cancel{\sec^2}\theta}$

$= 1-\ln 3-0+\int_{\arctan\frac{2}{3\sqrt{3}}}^{\arctan\frac{8}{3\sqrt{3}}} \frac{2\sqrt{3}}{3}d\theta$

$= 1-\ln 3 + \frac{2\sqrt{3}}{3}\theta\Big|_{\theta=\arctan\frac{2}{3\sqrt{3}}}^{\theta=\arctan\frac{8}{3\sqrt{3}}}$

$= \boxed{\frac{2\sqrt{3}}{3}\left(\arctan\frac{8}{3\sqrt{3}}-\arctan\frac{2}{3\sqrt{3}}\right)}$

$$\begin{array}{r} 1-\frac{2x}{x^2+x+1} \\ (x^2+x+1)\overline{)\,x^2-x+0} \\ -(x^2+x) \\ \hline -2x+0 \end{array}$$

$\int\frac{2x+1}{x^2+x+1}dx$

$u=x^2+x+1$

$\frac{du}{dx}=2x+1$

$dx=\frac{du}{2x+1}$

$=\int\frac{\cancel{2x+1}}{u}\cdot\frac{du}{\cancel{2x+1}}$

$=\int\frac{du}{u}$

$=\ln|u|+C$

$x^2+x+1 = x^2+x+(\frac{1}{2})^2+1-(\frac{1}{2})^2$

$x^2+x+1 = (x+\frac{1}{2})^2+\frac{3}{4}$

So...

$x^2+x+1 = \left(\sqrt{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right)^2$

$x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$

$\frac{dx}{d\theta}=\frac{\sqrt{3}}{2}\sec^2\theta$

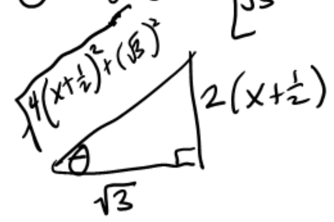
$dx=\frac{\sqrt{3}}{2}\sec^2\theta d\theta$

$x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$

$\tan\theta=\frac{2}{\sqrt{3}}(x+\frac{1}{2})$

$\theta=\arctan\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right]$

*



$* \sqrt{\frac{4(x+\frac{1}{2})^2}{4}+\frac{(\sqrt{3})^2}{4}}$

$= \sqrt{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$ see below and $\left(\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)^2 = \left(\sqrt{\left(\frac{\sqrt{3}}{2}\tan\theta\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)^2$

$$= \left(\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2\tan^2\theta+\left(\frac{\sqrt{3}}{2}\right)^2}\right)^2$$

$$= \left(\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2\left(\tan^2\theta+1\right)}\right)^2$$

$$= \left(\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2\sec^2\theta}\right)^2$$

$$= \frac{3}{4}\sec^2\theta$$

limits:

upper → $1+\frac{1}{3} = \frac{\sqrt{3}}{2}\tan\theta$

$\frac{2}{\sqrt{3}}\cdot\frac{4}{3} = \tan\theta$

$\frac{8}{3\sqrt{3}} = \tan\theta$

and

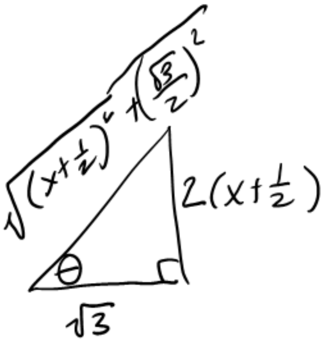
$\theta = \arctan\frac{8}{9}$ → QII

lower → $0+\frac{1}{3} = \frac{\sqrt{3}}{2}\tan\theta$

$\frac{2}{3\sqrt{3}} = \tan\theta$

$\frac{2\sqrt{3}}{9} = \tan\theta$

$\theta = \arctan\left(\frac{2\sqrt{3}}{9}\right)$



$\tan\theta = \frac{2(x+\frac{1}{2})}{\sqrt{3}}$

$x=1$:

$\tan\theta = \frac{2(4/3)}{\sqrt{3}}$

$\tan = \frac{8}{3\sqrt{3}}$

8. $\int_0^2 x^2 e^{-2x} dx$

$$\int x^2 e^{-2x}\, dx = (x^2)\left(-\frac{1}{2}e^{-2x}\right) - \int\left(-\frac{1}{2}e^{-2x}\right)(2x\, dx)$$

$$= -\frac{1}{2}x^2 e^{-2x} + \int x e^{-2x}\, dx$$

$$= -\frac{1}{2}x^2 e^{-2x} + \left[x\left(-\frac{1}{2}e^{-2x}\right) - \int -\frac{1}{2}e^{-2x}\, dx\right]$$

$$= -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} + \frac{1}{2}\int e^{-2x}\, dx$$

$$= -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} + \frac{1}{2}\left(\frac{e^{-2x}}{-2}\right) + C$$

$$= \boxed{-\frac{1}{4}e^{-2x}\left(2x^2 + 2x + 1\right) + C}$$

1st time IBP

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\int dv = \int e^{-2x} dx$

$v = \frac{e^{-2x}}{-2}$

2nd time IBP

$u = x$

$\frac{du}{dx} = 1$

$du = dx$

$\int dv = \int e^{-2x} dx$

$v = \frac{e^{-2x}}{-2}$

9. $\int \frac{\sin x}{\cos x + \cos^2 x} dx$

$$= \int \frac{\cancel{\sin x}}{u + u^2} \cdot \frac{du}{-\cancel{\sin x}}$$

$$= -\int \frac{1}{u^2 + u} \, du$$

$$= -\int \left(\frac{1}{u} - \frac{1}{u+1} \right) du$$

$$= -\ln|u| + \ln|u+1| + C$$

$$= \ln\left|\frac{u+1}{u}\right| + C$$

$$\boxed{\begin{gathered} = \ln\left|\frac{\cos x + 1}{\cos x}\right| + C \\ \text{OR} \\ = \ln|1 + \sec x| + C \end{gathered}}$$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

PFD

$$\frac{1}{u(u+1)} = \frac{A_1}{u} + \frac{A_2}{u+1}$$

$$\frac{1}{u(u+1)} = \frac{A_1(u+1) + A_2(u)}{u(u+1)}$$

$$0u^1 + 1u^0 = A_1 u + A_1 + A_2 u^1$$

$$0u^1 + 1u^0 = (A_1 + A_2)u^1 + A_1 u^0$$

$$A_1 + A_2 = 0 \rightarrow A_2 = -1$$

$$A_1 = 1$$

10. $\int \frac{xe^{3x}}{(3x+1)^2}dx$

$$= \left(xe^{3x}\right)\left(-\frac{1}{3(3x+1)}\right) - \int -\frac{e^{3x}\cancel{(3x+1)}dx}{3\cancel{(3x+1)}}$$

$$= -\frac{xe^{3x}}{3(3x+1)} + \frac{1}{3}\int e^{3x}dx$$

$$= -\frac{xe^{3x}\cdot 3}{3(3x+1)\cdot 3} + \frac{1}{3}\cdot\frac{e^{3x}\cdot(3x+1)}{3(3x+1)} + C$$

$$= \frac{-3\cancel{xe^{3x}} + 3\cancel{xe^{3x}} + e^{3x}}{9(3x+1)} + C$$

$$= \boxed{\frac{e^{3x}}{9(3x+1)} + C}$$

$u = xe^{3x}$

$\frac{du}{dx} = 3xe^{3x} + e^{3x}$

$du = e^{3x}(3x+1)dx$

$\int dv = \int (3x+1)^{-2}dx$

$v = \frac{1}{3}\frac{(3x+1)^{-1}}{-1}$

$v = -\frac{1}{3(3x+1)}$

Evaluate the following limits. EXACT ANSWERS ONLY PLEASE!!!

1. $\lim_{x\to 0^+} x^x \overset{D.S.}{=} 0^0$ indeterminate

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x}$$

$$= \lim_{x\to 0^+} e^{x\ln x}$$

$$= \lim_{x\to 0^+} e^{\frac{\ln x}{\left(\frac{1}{x}\right)}}$$

$$= e^{\lim_{x\to 0^+} \frac{\ln x}{1/x}}$$

$$= e^{\lim_{x\to 0^+} \frac{1/x}{-1/x^2}}$$

$$= e^{\lim_{x\to 0^+}(-x)}$$

$$= e^{-0}$$

$$= \boxed{1}$$

2. $\lim_{x\to 0^+} \sin x \ln x = 0\cdot(-\infty)$ indeterminant

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\sin x}{\frac{1}{\ln x}}$$

$$= \frac{0}{0}$$

So...

$$\lim_{x\to 0^+} \frac{\sin x}{\frac{1}{\ln x}} = \lim_{x\to 0^+} \frac{\cos x}{-\frac{\frac{1}{x}}{(\ln x)^2}}$$

$$= \lim_{x\to 0^+} \frac{\cos x}{\left[\frac{1}{x}\cdot\frac{1}{\ln x}\right]}$$

$$= \frac{\cos 0}{\frac{1}{\infty}\cdot\frac{1}{-\infty}}$$

$$= \frac{1}{0}$$ so $\boxed{\lim_{x\to 0^+} \sin x \ln x \text{ DNE}}$

3. $\lim_{x\to\infty}(x-\ln x) = \infty-\infty$

$$\lim_{x\to\infty}(x-\ln x) = \lim_{x\to\infty} \ln\left[e^{(x-\ln x)}\right]$$

$$= \lim_{x\to\infty} \ln\frac{e^x}{e^{\ln x}}$$

Now... $\lim_{x\to\infty}\frac{e^x}{x}$

$$= \frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\frac{e^x}{x} \overset{LR}{=} \lim_{x\to\infty}\frac{e^x}{1}$$

$$= \infty$$

and $\ln(\infty)\to\infty$

$\boxed{\text{DNE}}$

Find the area of the region bounded by $f(x)=\sqrt{x^2+4}$, $y=0$ $x=1$, and. $x=4$.

$$A=\int_1^4 (x^2+4)^{1/2}\,dx$$

$$A=\int_{\arctan\frac{1}{2}}^{\arctan 2} (2\sec\theta)(2\sec^2\theta\, d\theta)$$

$$A=4\underbrace{\int_{\arctan\frac{1}{2}}^{\arctan 2} \sec^3\theta\, d\theta}$$

$$A=4\left[\underbrace{\frac{1}{2}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right.}\right]_{\theta=\arctan\frac{1}{2}}^{\theta=\arctan 2}$$

$$A=2\left[\sec(\arctan 2)\tan(\arctan 2)+\ln\left|\sec(\arctan 2)+\tan(\arctan 2)\right|\right]$$

$$-2\left[\sec(\arctan\tfrac{1}{2})\tan(\arctan\tfrac{1}{2})+\ln\left|\sec(\arctan\tfrac{1}{2})+\tan(\arctan\tfrac{1}{2})\right|\right]$$

$$=2\left[\sec(\arctan 2)(2)+\ln\left|\sec(\arctan 2)+2\right|\right]$$

$$-2\left[\sec(\arctan\tfrac{1}{2})(\tfrac{1}{2})+\ln\left|\sec(\arctan\tfrac{1}{2})+\tfrac{1}{2}\right|\right]$$

$$=\boxed{2\left[2\sec(\arctan 2)+\ln\left|\frac{\sec(\arctan 2)+2}{\sec(\arctan 1/2)+\frac{1}{2}}\right|-\frac{1}{2}\sec(\arctan\tfrac{1}{2})\right]}$$

$$\int\sec^3\theta\, d\theta=\int\sec\theta\sec^2\theta\, d\theta$$

$$\int\sec^3\theta\, d\theta=\sec\theta\tan\theta-\int(\tan\theta)(\sec\theta\tan\theta\, d\theta)$$

$$\int\sec^3\theta\, d\theta=\sec\theta\tan\theta-\int\tan^2\theta\sec\theta\, d\theta$$

$$\int\sec^3\theta\, d\theta=\sec\theta\tan\theta-\int(\sec^2\theta-1)\sec\theta\, d\theta$$

$$\int\sec^3\theta\, d\theta=\sec\theta\tan\theta-\int\sec^3\theta\, d\theta+\int\sec\theta\, d\theta$$

$$\int\sec^3\theta\, d\theta=\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|+C$$

$$2\int\sec^3\theta\, d\theta=\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|+C$$

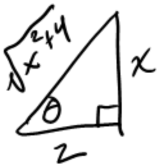
$$\int\sec^3\theta\, d\theta=\frac{1}{2}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|+C\right.$$

$x=2\tan\theta$

$dx=2\sec^2\theta\, d\theta$

$\tan\theta=\frac{x}{2}$

$\theta=\arctan\frac{x}{2}$



upper limit:

$4=2\tan\theta$

$2=\tan\theta$

$\arctan 2=\theta$

lower limit:

$1=2\tan\theta$

$\frac{1}{2}=\tan\theta$

$\arctan\frac{1}{2}=\theta$

$$\sqrt{x^2+4}=\sqrt{(2\tan\theta)^2+4}$$

$$=\sqrt{4\tan^2\theta+4}$$

$$=\sqrt{4(\tan^2\theta+1)}$$

$$=2\sqrt{\sec^2\theta}$$

$$=2\sec\theta$$

$u=\sec\theta$

$du=\sec\theta\tan\theta\, d\theta$

$\int dv=\int\sec^2\theta\, d\theta$

$v=\tan\theta$

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m - n)x] - \cos[(m + n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m - n)x] + \sin[(m + n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m - n)x] + \cos[(m + n)x])$$